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THE UNIFORMIZATION PROPERTY FOR $\aleph_2$

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ABSTRACT

We present S. Shelah's result that $S_1^2 = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_1\}$ may have the uniformization property (cf., §1, or [3] for a definition) for "well-chosen sequences", $\langle \eta_\delta : \delta \in S_1^2 \wedge \eta_\delta$ an increasing ω_1-sequence of ordinals converging to $\delta\rangle$. This implies that GCH $\not\rightarrow \diamondsuit_{S_1^2}$, which shows that Gregory's result (cf., [2]), GCH $\rightarrow \diamondsuit_{S_0^2}$, is the best possible.

§1. Introduction

In this paper, we present the proof of a generalization, due to S. Shelah, of a result proved in [3]. Following the terminology employed in [3], we say that $S \subseteq \omega_1$ has the *uniformization property* if for every sequence $\langle \eta_\delta : \delta \in S\rangle$, where each η_δ is an increasing ω-sequence converging to δ, and for each $c = \langle c_\delta : \delta \in S \wedge c_\delta \in {}^{\omega}2\rangle$ (referred to as a coloring of $\langle \eta_\delta : \delta \in S\rangle$), there exists an $f : \omega_1 \rightarrow 2$ so that for each $\delta \in S$ there is an $m_\delta < \omega$ such that $(\forall m > m_\delta) f(\eta_\delta(m)) = c_\delta(m)$ (i.e., f uniformizes the coloring). In [3], the following is proved:

(1.1) Con(ZF) $\rightarrow$ Con(ZFC + CH + "Some stationary $S \subseteq \omega_1$ has the uniformization property").

There, variations of this consistency proof yielding the consistency with CH of like combinatorial principles are mentioned, and applications are discussed. The uniformization property for subsets of ω_1 is dealt with in [1] also. It is proved there, under CH, that no closed, unbounded subset of ω_1 can have the uniformization property, and also that for $S \subseteq \omega_1$, if $\Phi(S)$ holds, then S cannot have the uniformization property ($\Phi(S)$, introduced and treated extensively in [1], is a weak form of $\diamondsuit$.)

Interest in a generalization of (1.1) to subsets of ω_2 arises from two directions. First, one simply might want to determine what analogues of (1.1) hold in higher

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cardinalities. The second, and by far more interesting reason has to do with the fact that certain versions of $\diamond$ for cardinalities higher than ω_1 have been shown to follow from the G.C.H., and as alluded to above, it seems that $\diamond$-like principles have something to do with the uniformization property. Thus, relative consistency results involving the uniformization property might clarify the status of some of these versions of $\diamond$. Let us be more precise now. Gregory, in [2], has shown that $\diamond_{S^2_0}$ (throughout this paper, $S^\alpha_\beta = \{\delta < \omega_\alpha : \mathrm{cf}(\delta) = \omega_\beta\}$) follows from the G.C.H. Furthermore, $\diamond_{S^2_0}$ easily implies that S^2_0 cannot have the uniformization property. (This, and the same assertion for stationary subsets of other S^{α}_β's follows by a completely straightforward adaption of the proofs given in [1] which establish that $\diamond$ implies Φ ([1], p. 237) and that Φ implies the negation of the uniformization property for S^1_0 ([1], theorem 5.1).) G.C.H. therefore implies that S^2_0 cannot have the uniformization property. Consequently, information about the relative consistency of the G.C.H. and the uniformization property for S^2_1 (i.e., the statement that increasing ω_1-sequences converging to $\delta \in S^2_1$ are to be uniformized by some $f \in {}^{\omega_2}2$) might help to resolve the status of $\diamond_{S^2_1}$ and the G.C.H.

Indeed, the natural generalization of (1.1) to ω_2 and stationary subsets $S \subseteq S^2_1$ can be proved by appropriately modifying the proof of (1.1) (cf., [3], theorem 2.3). In this case, however, if $S \subseteq S^2_1$ is to have the uniformization property, then $S^2_1 - S$ must be stationary as well. So, the more important question is whether or not S^2_1 itself can have the uniformization property. The primary result of this paper is:

THEOREM 2.1. $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH} + S^2_1$ *has the uniformization property for a "well-chosen sequence"* $\langle \eta_\delta : \delta \in S^2_1 \rangle)$.

Nb. The meaning of "well-chosen sequence" will be made precise in 2.7.

This theorem thus shows that Gregory's result cannot be improved.

All results and methods of proof are due to S. Shelah, to whom the authors wish to express gratitude for patient explanations and encouragement. We also wish to thank Arnold Miller for many helpful conversations.

Throughout, we use the usual notation and conventions.

§2. In this section, we will prove the main theorem of this paper:

THEOREM 2.1. $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH} + S^2_1$ *has the uniformization property for a "well-chosen sequence"* $\langle \eta_\delta : \delta \in S^2_1 \wedge \eta_\delta : \omega_1 \to \delta \wedge \eta_\delta$ *is increasing* $\wedge \sup_{\alpha<\omega_1}\{\eta_\delta(\alpha)\} = \delta \rangle$.) (See 2.7 for a definition of "well-chosen sequence".)

Theorem 2.1 follows from:

THEOREM 2.2. *Assume* $V = L$. *Let* $S_1^2 = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_1\}$. *Then, for a "well-chosen sequence",* $\langle \eta_\delta : \delta \in S_1^2 \rangle$, *there is a set of forcing conditions* $\mathbf{P} = \langle P, \leqq \rangle$ *such that*:

(1) $|P| = \aleph_3$, $\mathbf{P}$ *has the* $\aleph_3$ *chain condition, and* $\mathbf{P}$ *adds no new sequences of length* ω *or* ω_1.

(2) *In the generic extension* $V^{\mathbf{P}}$, *for any* $\bar{c} = \langle c_\delta : \delta \in S_1^2 \wedge c_\delta \in {}^{\omega_1}2 \rangle$, *there is a function* $f : \omega_2 \to 2$ *so that* $(\forall \delta \in S_1^2)(\exists \nu_\delta < \omega_1)(\forall \nu < \omega_1)[\nu > \nu_\delta \to f(\eta_\delta(\nu)) = c_\delta(\nu)]$. (I.e., for the "well-chosen sequence" $\langle \eta_\delta : \delta \in S_1^2 \rangle$, the uniformization property holds in $V^{\mathbf{P}}$ for S_1^2.)

Before presenting the proof of Theorem 2.2, which uses iterated forcing, we briefly sketch the first step in the iteration for the benefit of the reader.

Let $\bar{c} = \langle c_\delta : \delta \in S_1^2 \wedge c_\delta \in {}^{\omega_1}2 \rangle$ be given. We then define $\mathbf{P} = \langle P_{\bar{c}}, \leqq \rangle$ as follows: $P_{\bar{c}} = \{f : f \in {}^{\alpha}2 \quad (\alpha < \omega_2) \wedge (\forall \delta \leqq \alpha) \quad [\delta \in S_1^2 \to (\exists \xi_\delta < \omega_1)(\forall \xi > \xi_\delta) f(\eta_\delta(\xi)) = c_\delta(\xi)]\}$, and for $p, q \in P_{\bar{c}}$, $p \leqq q$ if and only if $p \subseteq q$. It is apparent that $E_\alpha = \{p : \alpha \subseteq \mathrm{dom}(p)\}$ is dense for all $\alpha < \omega_2$, whereupon it follows that a generic filter will yield the desired unifying function. Moreover, $\mathbf{P}_{\bar{c}}$ does not add new ω-sequences since the union of a countable chain of conditions in $\mathbf{P}_{\bar{c}}$ is itself a member of $\mathbf{P}_{\bar{c}}$ (this is because each η_δ converges to δ, which has cofinality ω_1). Also, since $|\mathbf{P}_{\bar{c}}| = \aleph_2$, $\mathbf{P}_{\bar{c}}$ clearly satisfies the $\aleph_3$-chain condition.

Consequently, in order for us to see that the generic extension preserves cardinals and the G.C.H., it only remains to indicate why no new ω_1-sequences are added. Thus, suppose that $p \Vdash$ "τ is an ω_1-sequence". We must find some $q \geqq p$ that decides τ (i.e., $q \Vdash \tau = \check{f}$, for some $f \in V$). To this end, choose $N \prec \langle H(\omega_2), \mathbf{P}_{\bar{c}}, \Vdash, p, \bar{c}, \tau, \cdots \rangle$ with $N \cap \omega_2 = \delta \in S_1^2$ and $N = \bigcup_{\xi < \omega_1} N_\xi$, where $\langle N_\xi : \xi < \omega_1 \rangle$ forms a continuous, increasing elementary chain, $\omega_1 \subseteq N_0$, $N_\xi \cap \omega_2 = \delta_\xi$, and $(\forall \xi < \omega_1)^{\omega} N_{\xi+1} \subseteq N_{\xi+1}$. We define, by induction, an increasing chain of conditions $p_\xi \in N_\xi$ for each $\xi < \omega_1$ so that p_ξ decides $\tau \restriction \xi$. Let $p_0 = p$. At limits λ, we let $p_\lambda = \bigcup_{\xi < \lambda} p_\xi$ ($\mathrm{cf}(\lambda) = \omega$, so p_λ indeed is a condition). The only work comes at successor steps. Assume, then, that we have defined p_ξ and wish to extend it to $p_{\xi+1}$ which fulfills the criteria stated above. Note that $\{\beta : \beta \in \mathrm{rng}(\eta_\delta) \cap N_{\xi+1} \setminus N_\xi\}$ is countable. In $N_{\xi+1}$, we form a function $f = p_\xi \cup \{\langle \eta_\delta(\nu), c_\delta(\nu) \rangle : \delta_\xi \leqq \eta_\delta(\nu) < \delta_{\xi+1}\}$. Since $|\mathrm{dom}(f) \setminus \mathrm{dom}(p_\xi)| \leqq \aleph_0$, we may find a condition $r \in N_{\xi+1}$ that extends f. Then, again in $N_{\xi+1}$, we obtain $p_{\xi+1} \geqq r$ such that $p_{\xi+1}$ decides $\tau \restriction \xi + 1$. Finally it is easy to see that $\bigcup_{\xi < \omega_1} p_\xi = q$ is a condition extending p which decides τ, completing the argument. Hence $\mathbf{P}_{\bar{c}}$ has the properties that it should have.

The idea underlying the proof of Theorem 2.2 is to iterate $\mathbf{P}_{\bar{c}}$ extensions ω_3 times, taking inverse limits at limit stages of cofinality less than ω_2, and direct limits at limit stages of cofinality ω_2. As will be seen, much more care must be exercised in the iteration than in the sketch of the first step just given.

Now, we inductively define sets of forcing conditions, $\mathbf{P}_\alpha$, and names, $\bar{c}_\alpha$, in the $\mathbf{P}_\alpha$-forcing language, for $\alpha < \omega_3$. $\mathbf{P} = \mathbf{P}_{\omega_3}$ will be the desired set of conditions.

Definition 2.3. $P_\alpha = \{p : p$ is a function $\wedge \operatorname{dom}(p) \subseteq \alpha \wedge |\operatorname{dom}(p)| = \aleph_1 \wedge (\forall \beta \in \operatorname{dom}(p))\ [p \restriction \beta \Vdash^{\mathbf{P}_\beta}$ "$p(\beta)$ is a name in the $\mathbf{P}_\beta$ forcing language for a function from $\gamma^\beta_p < \omega_2$ to 2 such that $p(\beta) \in \mathbf{P}_{\bar{c}_\beta}$"$] \wedge \langle \gamma^\beta_p : \beta \in \operatorname{dom}(p) \rangle \in V\}$. P_α will be ordered by $p \leqq q$ (i.e., q extends p) $\Leftrightarrow (\forall \beta \in \operatorname{dom}(q))\ q \restriction \beta \Vdash^{\mathbf{P}_\beta} p(\beta) \subseteq q(\beta)$.

We shall say that $p \in \mathbf{P}_\alpha = \langle P_\alpha, \leqq \rangle$ is *real* if $(\forall \beta \in \operatorname{dom}(p))\ (\exists f \in V)\ p \restriction \beta \Vdash^{\mathbf{P}_\beta}$ "$p(\beta) = \check{f}$", and we shall call p *δ-rectangular* if $(\forall \beta \in \operatorname{dom}(p)) \gamma^\beta_p = \delta$.

Definition 2.4. If $p \in \mathbf{P}_\alpha$ and $q \in \mathbf{P}_\beta$, then $p \vee q$ is the relation defined on $\operatorname{dom}(p) \cup \operatorname{dom}(q)$ so that:

(i) if $\xi \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)$, then $[p \vee q](\xi) = p(\xi)$.

(ii) if $\xi \in \operatorname{dom}(q) \setminus \operatorname{dom}(p)$, then $[p \vee q](\xi) = q(\xi)$.

(iii) if $\xi \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$, then $[p \vee q](\xi) = p(\xi) \cup q(\xi)$.

Proposition 2.5. *If $p \in \mathbf{P}_\alpha$, $q \in \mathbf{P}_\beta$ and $\alpha < \beta$, then $p \geqq q \restriction \alpha$ implies that $p \vee q \in \mathbf{P}_\beta$.*

Proof. Left to the reader. ⊣

Lemma 2.6. *$\mathbf{P} = \mathbf{P}_{\omega_3}$ is $\aleph_1$-closed.*

Proof. This is clear since the union of countably many conditions cannot reach a new η_δ sequence. ⊣

Theorem 2.2 now will follow from Lemma 2.8, below. However, before stating and proving the lemma, we must make precise what we mean by a "well-chosen sequence", $\langle \eta_\delta : \delta \in S^2_1 \rangle$.

2.7. *Well-Chosen Sequences.* Recall that $V = L$ for Theorem 2.2. We shall employ $\diamondsuit_{S^2_1}$ to choose the η_δ's. Since $\diamondsuit_{S^2_1}$ holds, for every $\delta \in S^2_1$, there is a model M_δ for a countable language so that M_δ has universe δ and:

(2.7.1) $\{\delta \in S^2_1 : M_\delta \prec M\}$ is stationary, where M is a model, with universe ω_2, for a countable language.

Now we define, for each $\delta \in S^2_1$, η_δ, an ω_1-sequence converging to δ:

Case I. For some partial ordering $\mathbf{P} \in H(\omega_3), \Vdash^{\mathbf{P}}, \cdots$, and for some structure M with universe ω_2, the following hold:

(2.7.2)(a) $M \prec \langle H(\omega_3), \mathbf{P}, \Vdash^{\mathbf{P}}, \cdots, ES\rangle$, where

$$ES = \{N \in H(\omega_3) : \bar{\bar{N}} = \aleph_1 \wedge {}^{\omega}N \subseteq N \wedge N \prec \langle H(\omega_3), \mathbf{P}, \Vdash^{\mathbf{P}}, \cdots\rangle$$
$$\wedge\, N \cap \omega_2 \text{ is an ordinal}\} \quad (\text{i.e., } ES \subseteq H(\omega_3)).$$

(2.7.2)(b) $\mathrm{ORD} \cap \mathrm{ord}(M) \supseteq \omega_2$.

(2.7.2)(c) $M_\delta \prec \langle M, \varepsilon, f\rangle$, where ε is the *real* ε (recall that the universe of M, $|M|$, is ω_2) and $f : \langle \omega_2^M, \varepsilon^M\rangle \to \langle |M|, \varepsilon\rangle$ is an order preserving bijection.

(2.7.2)(d) (transitive collapse of $M_\delta) \cap \omega_2 = \delta$.

Let $\langle \delta^0_\xi : \xi < \omega_1\rangle$ be an increasing, continuous sequence of ordinals converging to δ. We shall abuse notation slightly in what follows by confusing M_δ with its transitive collapse. By induction on $\xi < \omega_1$, we choose $a_\xi \in M_\delta$ so that: $(\forall \xi) M_\delta \models ES(a_\xi)$; for $\xi_1 < \xi_2$, $M_\delta \models a_{\xi_1} \in a_{\xi_2}$ (ε here is ε^{M_δ}); for λ a limit, $M_\delta \models \bigcup_{\xi<\lambda} a_\xi = a_\lambda$; and $|a_{\xi+1}| \cap (\mathrm{ord}(M_\delta) \cap \omega_2) \supseteq \delta^0_\xi$ (where $|a_{\xi+1}|$ is the universe of the structure $a_{\xi+1}$). For $\xi < \omega_1$, let $\delta_\xi = |a_\xi| \cap \omega_2$. In this case we let $\mathrm{rng}(\eta_\delta) = \{\delta_\xi : \xi \notin S\}$, where $S \subseteq \omega_1$ is stationary and costationary.

Case II. Not Case I. Let η_δ be any ω_1-sequence converging to δ.

This done, we shall establish Theorem 2.2 by proving the following lemma:

LEMMA 2.8. *For each* $\alpha \leqq \omega_3$,

(i) $(\forall p \in \mathbf{P}_\alpha)$ ($\forall$ *countable* $t \in {}^{\alpha\times\omega_2}2$) $[(\forall \beta < \alpha) p \restriction \beta \Vdash$ "$p(\beta)$ *and* $t(\beta)$ *are consistent*" $\Rightarrow$ $(\exists q \in \mathbf{P}_\alpha)$ $(p \leqq q \wedge (\forall \beta < \alpha) \varnothing \Vdash$ "$t(\beta) \subseteq q(\beta)$")$]$.

(ii) $(\forall p \in \mathbf{P}_\alpha)$ $(\exists q \in \mathbf{P}_\alpha)$ $[p \leqq q \wedge q$ *is real and rectangular*$]$.

(iii) *Forcing with* $\mathbf{P}_\alpha$ *adds no new* ω_1*-sequences.*

COROLLARY 2.9. $\mathbf{P} = \mathbf{P}_{\omega_3}$ *has the* $\aleph_3$*-chain condition.*

PROOF. Given $\aleph_3$ conditions, extend each to a real condition by 2.8(ii). GCH and a standard Δ-system argument implies that this set of conditions could not form an antichain. ⊣

Thus, assuming the lemma, we see that forcing with $\mathbf{P}$ does not change cardinals or cofinalities (merely combine 2.8(iii) and 2.9). That $\mathbf{P}$ satisfies all the other criteria required for Theorem 2.2 should be clear. So, it only remains to prove lemma 2.8.

PROOF OF 2.8. We shall prove (i)–(iii) by simultaneous induction on α. First, we will give the arguments for (i) and (ii), and then shall indicate how (ii) and a trivial modification of the argument used to establish (ii) in the case that $\mathrm{cf}(\alpha) = \omega_1$ suffices to yield (iii) for any α. Tacit use of Proposition 2.5 will be made throughout what follows.

$\alpha = 0$: There is nothing to do. $\dashv$

$\mathrm{cf}(\alpha) = \omega$: (i) and (ii) follow since $\mathbf{P}_\alpha$ is $\aleph_1$-closed. $\dashv$

$\mathrm{cf}(\alpha) > \omega_1$: (i) is obvious again, as t is countable, and (ii) follows because for any $p \in \mathbf{P}_\alpha$, $|\mathrm{dom}(p)| = \aleph_1$, whence $\sup(\{\beta : \beta \in \mathrm{dom}(p)\}) < \alpha$. $\dashv$

$\alpha = \beta + 1$: (i) This is easy, and left to the reader. $\dashv$

(ii) Given $p \in \mathbf{P}_\alpha$, we use the induction hypothesis to define $\langle q_n : n < \omega \rangle$ and $\langle \delta_n : n < \omega \rangle$ satisfying:

(1) $(\forall n < \omega)\delta_n < \delta_{n+1} < \omega_2$.

(2) $(\forall n < \omega)$ $[q_n \in \mathbf{P}_\beta \wedge q_n \leqq q_{n+1} \wedge q_n$ is real, δ_n-rectangular$]$.

(3) $q_0 \Vdash$ "$p(\beta) = \check{f}$", for some $f \in V$.

(4) $q_{n+1} \Vdash$ "$\langle \underset{\sim}{c}^\xi_\beta : \xi < \delta_n \rangle = \langle \check{c}^\xi_\beta : \xi < \delta_n \rangle$" for some $\langle c^\xi_\beta : \xi < \delta_n \rangle \in V$.

Then, let $q = \bigcup_{n<\omega} q_n$ and $\delta = \bigcup_{n<\omega} \delta_n$. In V, find $g : \delta \to 2$ uniformizing the colorings given by q and extending f. If $\check{g}$ is a $\mathbf{P}_\beta$ name for g (i.e., $\varnothing \Vdash^{\mathbf{P}_\beta} \check{g} : \delta \to 2$), then $q \cup \{\langle \alpha, \check{g} \rangle\}$ is the desired condition. $\dashv$

$\mathrm{cf}(\alpha) = \omega_1$: (i) Easy once again, and left to the reader. $\dashv$

(ii) This argument represents the bulk of the proof of the lemma. We begin by showing that it suffices to prove:

(ii)$'$ $\cdots (\forall p \in \mathbf{P}_\alpha)$ $(\exists q \in \mathbf{P}_\alpha)$ $[q \geqq p \wedge q$ is rectangular $\wedge (\forall \beta < \alpha)$ $(\forall r \in \mathbf{P}_\beta)$ $(r \geqq q \restriction \beta \wedge r \Vdash$ "$q \restriction \beta = \langle \check{f}_\gamma : \gamma < \beta \rangle$", for some $\langle f_\gamma : \gamma < \beta \rangle \in V \Rightarrow r \Vdash$ "$p(\beta) = \check{f}$", for some $f \in V)]$.

To see this, we first choose $q_n \in \mathbf{P}_\alpha$, for each $n < \omega$, by induction. Let $q_0 = p$. Given q_n, let q_{n+1} be as given by (ii)$'$, with q_n assuming the role of p. Let $q = \bigcup_{n<\omega} q_n$. Clearly, q is a condition extending p, and q is rectangular. We show by induction on $\beta < \alpha$ that q is real. Thus, suppose that $q \restriction \beta$ is real. We must prove that $q \restriction \beta \Vdash$ "$q(\beta) = \check{f}$" for some $f \in V$. Now, $q \restriction \beta \Vdash$ "$q(\beta) = \bigcup_n q_n(\beta)$", and since $q \restriction \beta$ is real, for each $n < \omega$, $q \restriction \beta \Vdash$ "$q_{n+1} \restriction \beta = \langle \check{f}^{n+1}_\gamma : \gamma < \beta \rangle$" for some $\langle f^{n+1}_\gamma : \gamma < \beta \rangle \in V$. Thus, by (ii)$'$ it follows that for all $n < \omega$, $q \restriction \beta \Vdash$ "$q_n(\beta) = \check{f}_n$", for some $f_n \in V$. Whence, it follows that $q \restriction \beta$ forces $q(\beta)$ to be the name for a function V, completing the argument.

Consequently, it only remains to prove (ii)$'$. It is here that we need the fact that $\langle \eta_\delta : \delta \in S^2_1 \rangle$ is "well-chosen", as in 2.7. Thus, we may assume that there are structures N_ξ $(\xi < \omega_1)$, so that for any ξ:

$N_\xi \prec \langle H(\omega_3), \varepsilon, \mathbf{P}_\alpha, \Vdash^{\mathbf{P}_\alpha}, \cdots \rangle$; $\|N_\xi\| = \aleph_1$; ${}^\omega N_{\xi+1} \subseteq N_{\xi+1}$; $\langle N_\xi : \xi < \omega_1 \rangle$ forms a continuous, increasing, elementary chain; and, if we let $\delta_\xi = N_\xi \cap \omega_2$ and $\delta = \bigcup_{\xi<\omega_1} \delta_\xi$, then there is a stationary $S \subseteq \omega_1$ so that $\xi \in S$ implies $[\delta_\xi, \delta_{\xi+1}) \cap \operatorname{rng}(\eta_\delta) = 0$.

We also may assume that the $p \in \mathbf{P}_\alpha$ given in (ii)′ is in N_0. (For otherwise, take the least $p \in \mathbf{P}_\alpha$, via some choice function that is put into $\langle H(\omega_3), \cdots \rangle$, for which (ii)′ fails, and thus being definable, $p \in N_0$. In that case, this entire argument would be by contradiction.) Let us also set $\bigcup_{\xi<\omega_1} N_\xi = N$, $A = \alpha \cap N = \{\alpha_\xi : \xi < \omega_1\}$ where we demand that $\alpha_\xi \in N_{\xi+1}$, and $A_\xi = \{\alpha_\eta : \eta < \xi\}$.

Let $T_\xi = \{f : \operatorname{dom}(f) = A_\xi \wedge (\forall \alpha_\zeta \in \operatorname{dom}(f)) f(\alpha_\zeta) : \operatorname{rng}(\eta_\delta) \cap [\delta_\zeta, \delta_\xi) \to 2\}$. By induction on $\xi < \omega_1$, we will define $Q^\xi = \{q^\xi_t : q^\xi_t \in \mathbf{P}_\alpha \wedge t \in T_\xi\}$ such that:

(2.9)
(a) $Q^\xi \in N_{\xi+1}$.
(b) $t_1 \restriction \beta \subseteq t_2 \restriction \beta \Rightarrow q^\xi_{t_1} \restriction \beta \subseteq q^\xi_{t_2} \restriction \beta$.
(c) $t \subseteq q^\xi_t$ (i.e., $\beta \in \operatorname{dom}(t) \Rightarrow \varnothing \Vdash^{\mathbf{P}_\alpha}$ "$t(\beta) \subseteq q^\xi_t(\beta)$").
(d) $\eta_1 < \eta_2 \wedge t_1 \in T_{\eta_1} \wedge t_2 \in T_{\eta_2} \wedge t_1 \subseteq t_2 \Rightarrow q^{\eta_1}_{t_1} \subseteq q^{\eta_2}_{t_2}$.
(e) $(\forall \beta \in \operatorname{dom}(q^\xi_t))\ \operatorname{dom}(q^\xi_t(\beta)) \subseteq \delta_\xi$.

Intuitively, T_ξ may be thought of as a tree of possible initial segments of $\langle c^\delta_\beta : \beta \in A_\xi \rangle$. Q^ξ thus is a set of conditions in $\mathbf{P}_\alpha$ each member of which is consistent with an element of T_ξ. Such a tree of approximations, or guesses, is needed here because we do not know what $\langle c^\delta_\beta : \delta \in A \rangle$ is. Now we proceed with the definition of Q^ξ.

$\xi = 0$: Let $q^0_\varnothing = p$.

ξ a limit: Just take inverse limits.

$\xi = \eta + 1$ *and* $\operatorname{rng}(\eta_\delta) \cap (\delta_\xi - \delta_\eta] \neq 0$: We may assume that we have a choice function available (as per remarks made above). We note that for any $t \in T_\xi$ there is a unique $s_t \in T_\eta$ so that $s_t \subseteq t$. We work in N_ξ. For $\beta \in A_\xi$, we let $q^\xi_t(\beta)$ be the first $\mathbf{P}_\beta$-name σ so that $\varnothing \Vdash^{\mathbf{P}_\beta}$ "$q^\eta_{s_t}(\beta) \leqq \sigma \wedge \sigma \in \mathbf{P}_{c_\beta} \wedge t(\beta) \subseteq \sigma$". If $\beta \in \operatorname{dom}(q^\eta_s) \setminus A_\xi$, let $q^\xi_t(\beta) = q^\eta_t(\beta) = q^\eta_{s_t}(\beta)$. It is clear that q^ξ_t satisfies (2.9) (a)–(e).

$\xi = \eta + 1$ *and* $\operatorname{rng}(\eta_\delta) \cap (\delta_\xi - \delta_\eta] = 0$: In this case, $\eta \in S$, so $\diamondsuit_S$ gives us a $t^*_\eta \in T_\eta$ and a $\beta^*_\eta \in A_\eta$. Moreover, just as in the previous case, for each $t \in T_\xi$ there is a unique $s_t \in T_\eta$ with $s_t \subseteq t$. (In fact, since $\eta \in S$, $T_\eta = T_\xi$!) Now, by induction hypothesis (iii), we find $q_0 \in \mathbf{P}_{\beta^*_\eta}$, q_0 real and $q_0 \geqq q^\eta_{t^*_\eta} \restriction \beta^*_\eta$, so that $q_0 \Vdash$ "$q^\eta_{t^*_\eta}(\beta^*_\eta) = \check{f}$", for some $f \in V$. By the choice of N_ξ, we may assume that $f, q_0 \in N_\xi$. This done, we can define q^ξ_t for $t \in T_\xi$. Let $\gamma(t) = \min\{\beta^*_\eta, \gamma'(t)\}$, where $\gamma'(t) = \sup\{\zeta : t \restriction \zeta = t^*_\eta \restriction \zeta\}$. Then set $q^\xi_t = q_0 \restriction \gamma(t) \vee q^\eta_{s_t}$. Again, it is evident that (2.9) is satisfied.

This completes the definition of Q^ξ $(\xi < \omega_1)$. So now we will define q as required in (ii)′ to complete the proof of (ii)′. We must define q conditionally (i.e., we carry out a so-called fusion construction). Thus, for $\beta < \alpha$, given $q \restriction \beta$, we must define $q(\beta)$. Let $\{p_\gamma : \gamma < \omega_2\}$ be a maximal antichain of conditions in $\mathbf{P}_\beta$ so that for all $\gamma, p_\gamma \geqq q \restriction \beta$ and also for each $\eta \in A$, $\eta \leqq \beta$, $p_\gamma \Vdash^{\mathbf{P}_\beta}$ "$\underset{\sim}{c}{}^{\delta}_{\eta} = \check{c}^{\gamma}_{\eta}$" for some $c^\gamma_\eta \in V$. (Recall, by induction hypothesis, that forcing with $\mathbf{P}_\beta$ adds no new ω_1-sequences.) Then, let

$$q(\beta) = \left\{ \left\langle \left(\bigcup_{\xi < \omega_1} q^{\xi}_{t_\xi}(\beta) \right)^{\vee}, p_\gamma \right\rangle : t_\xi \restriction (\beta + 1) = \langle c^\gamma_\eta : \eta \in A, \eta \leqq \beta \rangle \restriction \delta_\xi \right\}.$$

The definition is unambiguous (i.e., beyond fulfilling the stated criteria, the choice of the t_ξ's doesn't matter) because of the conditions (2.9.1) (b)–(d). This done, we must show that (ii)′ indeed holds. Clearly, q is rectangular and it extends p. So, given $\beta < \alpha$, let $r \in \mathbf{P}_\beta$ be such that $r \geqq q \restriction \beta$ and $r \Vdash$ "$q \restriction \beta = \langle \check{f}_\gamma \rangle_{\gamma < \beta}$", for some $\langle f_\gamma \rangle_{\gamma < \beta} \in V$. If we let

$$t = \langle f_\gamma \rangle_{\gamma < \beta} \restriction \bigcup_{\alpha_\eta \in A} \{\langle \alpha_\eta, \zeta \rangle : \delta_\eta \leqq \zeta < \delta \wedge \zeta \in \operatorname{rng}(\eta_\delta)\},$$

then $\diamondsuit_S$ insures that there is a stationary set $S' \subseteq S$ so that for each $\xi \in S'$,

$$\beta = \beta^*_\xi \quad \text{and} \quad t \restriction \bigcup_{\alpha_\eta \in A_\xi} \{\langle \alpha_\eta, \zeta \rangle : \delta_\eta \leqq \zeta < \delta_\xi\} = t^*_\xi \restriction \beta.$$

Thus, $r \geqq q^{\xi}_{t_\xi} \restriction \beta$ and since $q^{\xi}_{t_\xi}(\beta) \supseteq p(\beta)$, we have that $r \Vdash^{\mathbf{P}_\beta}$ "$p(\beta) = \check{f}$" for some $f \in V$. This completes the proof of (ii)′.

Finally, to complete the proof of Lemma 2.8, we briefly indicate how the argument above may be modified to yield (iii), for any α. We assume that (i) and (ii) already have been established for α. So assume that $p \in \mathbf{P}_\alpha$ and $p \Vdash$ "τ is an ω_1-sequence". Essentially, we perform the construction to get $q \geqq p$ just as in the proof of (ii)′. Here, though, we need only ask $\diamondsuit_S$ to provide us with $t^*_\eta \in T_\eta$, for $\eta \in S$. Furthermore, at stage $\xi = \eta + 1$ in the construction involving $\diamondsuit_S$, we find (using the notation of the argument, above) $q_0 \in \mathbf{P}_\alpha$ with q_0 real and $q_0 \geqq q^{\eta}_{t^*_\eta}$, so that $q_0 \Vdash$ "$\tau \restriction \xi = \check{f}$" for some $f \in V$ (recall $\mathbf{P}_\alpha$ is $\aleph_1$-complete). Then, for $t \in T_\xi$ let $q^\xi_t = q_0 \restriction \gamma'(t) \vee q^{\eta}_{s_t}$ except when $t \supseteq t^*_\eta$, in which case let $q^\xi_t = q_0$. If q then is built as before, next find $r \in \mathbf{P}_\alpha$, $r \geqq q$ and r real. Applying $\diamondsuit_S$, it is easy to see that $r \Vdash$ "$\tau = \check{h}$", for some $h \in V$. This completes the proof of Lemma 2.8, and thus establishes Theorem 2.2. ⊣

We remark, in closing, that Theorem 2.1 may be generalized to successor cardinals by invoking the appropriate combinatorial principles, and to inaccessible cardinals as well. However, we do not know the status of Theorem 2.1 for less well-chosen sequences $\langle \eta_\delta : \delta \in S^2_1 \rangle$.

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